A new approach to limits?

Here's the problem: In college calculus books they always assume the limit of something before they solve what the limit is. The result is what you expected in the first place. So, how do you solve for the limit is the limit is *not* known in the first place? The following shows an approach to this problem that has some rather unexpected results.

Definition of the limit of a function
Let

$f : D \rightarrow \mathbb{R}$ be a function defined on a subset $D \subseteq \mathbb{R}$

c be a limit point of D

L be a real number. Then the statement

$$\forall \epsilon < 0 \ , \ \exists \delta > 0 \ :$$

$$\forall x \left(0 < |x - c| < \delta \Rightarrow |f(x) - L| < \epsilon \right)$$

is abbreviated to

$$\lim_{x \to c} f(x) = L$$

Less formally, "For all $\epsilon > 0$ there exists some $\delta > 0$ such that for all x in D that satisfy $0 < |x - c| < \delta$, the inequality $|f(x) - L| < \epsilon$ holds.

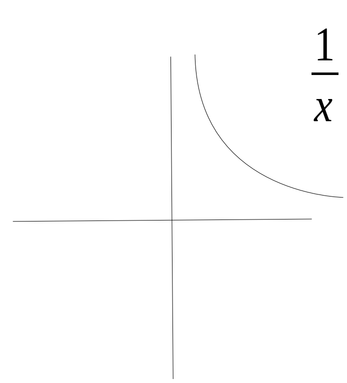
$\forall$ = "for all", e.g. $\forall P(x)$ means $P(x)$ is true for all x (universal quantification)

$\exists$ = "there exists", e.g. $\exists x : P(x)$ means there is at least one x such that $P(x)$ is true

(existential quantification)

example 1

$$\lim_{x \to \infty} \frac{1}{x} = L$$



$$|F(x) - L| < \varepsilon \Rightarrow L - \varepsilon < F(x) < L + \varepsilon$$
$$|x - x_L| < \delta \Rightarrow x_L - \delta < x < x_L + \delta \, , where \, x_L = \infty$$
$$\Rightarrow \infty - \delta < x < \infty + \delta$$
$$F(x) = \frac{1}{x}$$

$$-\varepsilon<\frac{1}{x}-L<+\varepsilon$$
$$-\varepsilon-\frac{1}{x}<-L<\varepsilon-\frac{1}{x}$$
$$-(-\varepsilon-\frac{1}{x})>L>-(\varepsilon-\frac{1}{x})$$
$$\frac{1}{x}+\varepsilon>L>\frac{1}{x}-\varepsilon$$
$$\infty-\delta<\infty+\delta\Rightarrow\frac{1}{\infty-\delta}>\frac{1}{\infty+\delta}$$

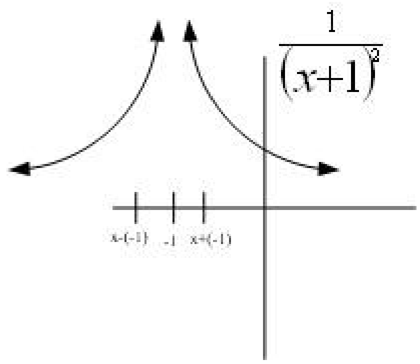
so

$$\frac{1}{\infty-\delta}+\varepsilon>L>\frac{1}{\infty+\delta}-\varepsilon\Rightarrow\frac{1}{\infty}>L>\frac{1}{\infty}\Rightarrow 0>L>0$$

Which means $L=0$

example 2

$$\lim_{x\to\infty}\frac{1}{(1+x)^2}=L$$



$$|F(x)-L|<\varepsilon\Rightarrow L-\varepsilon<F(x)<L+\varepsilon$$
$$|x-x_L|<\delta\Rightarrow x_L-\delta<x<x_L+\delta\,, where\ x_L=-1$$
$$\Rightarrow(-1)-\delta<x<(-1)+\delta$$
$$F(x)=\frac{1}{(1+x)^2}$$
$$L-\varepsilon<\frac{1}{(1+x)^2}<L+\varepsilon$$
$$\Rightarrow(L-\varepsilon)(1+x)^2<1<(L+\varepsilon)(1+x)^2$$
$$\Rightarrow(L-\varepsilon)(1+2\mathrm{x}+x^2)<1<(L+\varepsilon)(1+2\mathrm{x}+x^2)$$
$$\Rightarrow L-\varepsilon+2\mathrm{xL}-2\mathrm{x}\,\varepsilon+x^2L-x^2\varepsilon<1<L+\varepsilon+2\mathrm{xL}-2\mathrm{x}\,\varepsilon+x^2L-x^2\varepsilon$$
$$\Rightarrow L-\varepsilon+2\mathrm{xL}-2\mathrm{x}\,\varepsilon+x^2L-x^2\varepsilon<1$$
$$\Rightarrow L-\varepsilon+2(-1-\delta)L-2(-1-\delta)\varepsilon+(-1-\delta)^2L-(-1-\delta)^2\varepsilon<1$$
$$\Rightarrow L-\varepsilon-2L-2\delta L+2\varepsilon+2\delta\varepsilon+L+2\delta L+\delta^2L-\varepsilon-2\delta\varepsilon-\delta^2\varepsilon<1$$
$$\Rightarrow L-2\varepsilon-2L+2\varepsilon+L+\delta^2L-\delta^2\varepsilon<1$$
$$\Rightarrow\delta^2L-\delta^2\varepsilon<1$$
$$\Rightarrow\delta^2(L-\varepsilon)<1$$
$$\Rightarrow\delta^2<\frac{1}{L-\varepsilon}$$
$$as\,\delta\to 0,\varepsilon\to 0$$
$$\Rightarrow 0<\frac{1}{L-0}$$
$$\Rightarrow 0<\frac{1}{L}$$
$$\Rightarrow L<\infty$$

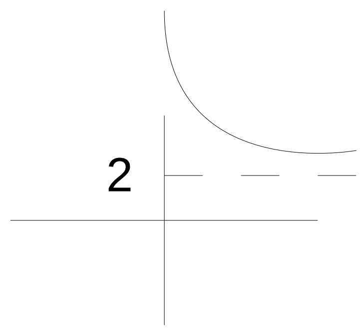
For

$$\frac{1}{(1+x)^2} < L+\varepsilon$$
$$1 < (L+\varepsilon)(1+2\mathrm{x}+x^2)$$
$$1 < L+\varepsilon+2\mathrm{xL}-2\mathrm{x}\,\varepsilon+x^2 L - x^2\,\varepsilon$$
$$\Rightarrow 1 < L+\varepsilon+2\mathrm{xL}+2\mathrm{x}\,\varepsilon+x^2 L+x^2\,\varepsilon < 1$$
$$\Rightarrow 1 < L+\varepsilon+2(-1+\delta)L+2(-1+\delta)\varepsilon+(-1+\delta)^2 L+(-1+\delta)^2\varepsilon$$
$$\Rightarrow 1 < L+\varepsilon-2L+2\delta L-2\varepsilon+2\delta\varepsilon+L-2\delta L+\delta^2 L+\varepsilon-2\delta\varepsilon+\delta^2\varepsilon$$
$$\Rightarrow 1 < \delta^2 L+\delta^2\varepsilon \Rightarrow 1 < \delta^2(L+\varepsilon)$$
$$\Rightarrow \frac{1}{\delta^2} < L+\varepsilon \Rightarrow \frac{1}{0} < L+0 \; as\, \delta \to 0, \varepsilon \to 0 \Rightarrow \infty < L$$

$$\Rightarrow \infty < L < \infty$$

Which means $L=\infty$

example 3



$$\lim_{x \to \infty} \frac{1}{x}+2=L$$
$$|F(x)-L| < \varepsilon \Rightarrow L-\varepsilon < F(x) < L+\varepsilon$$
$$|x-x_L| < \delta \Rightarrow x_L-\delta < x < x_L+\delta, where\, x_L=\infty$$
$$\Rightarrow \infty-\delta < x < \infty+\delta$$
$$F(x)=\frac{1}{x}+2$$
$$-\varepsilon < \frac{1}{x}+2-L < +\varepsilon$$
$$-\varepsilon-\left(\frac{1}{x}+2\right) < -L < \varepsilon-\left(\frac{1}{x}+2\right)$$
$$-\left(-\varepsilon-\left(\frac{1}{x}+2\right)\right) > L < -\left(\varepsilon-\left(\frac{1}{x}+2\right)\right)$$
$$\frac{1}{x}+2+\varepsilon > L > \frac{1}{x}+2-\varepsilon$$
$$\infty-\delta < \infty+\delta \Rightarrow \frac{1}{\infty-\delta} > \frac{1}{\infty+\delta}$$

so

$$\frac{1}{\infty-\delta}+2+\varepsilon > L > \frac{1}{\infty+\delta}+2-\varepsilon \Rightarrow \frac{1}{\infty}+2+0 > L > \frac{1}{\infty}+2+0 \Rightarrow 2 > L > 2$$

Which means $L=2$