This presentation was inspired in part by the Laws of Form by G. Spencer-Brown.

1. Arrangements Of Numbers

Numbers can be arranged in any way.

Example 1

6,7,8,256

Example 2:

1, 3, 5,

4, 5

6

Example 3:

356, 1000,

3, 456,

123 , 666

2. The Ordered Pair

An ordered pair is an arrangement of two numbers in which one precedes the other, hence the term “ordered”. For example,

1,0

0,1

223,356

An ordered pair is also an ordered twosome.

3. The Ordered Nsome

An ordered Nsome is an arrangement of N numbers in which each number precedes another save the last number, hence the term “ordered”. For example,

Example 1: An ordered foursome:

2,4,5,6

Example 2: An ordered sixsome

345,2,2.55,0.004,55,123

4. The One Dimensional Matrix

An ordered Nsome is a matrix, when an index added. The index can be anything, but is usually limited to numbers, especially natural numbers, i.e. 0, 1, 2, 3, …

Ordered pair 1, 0 as a matrix

index	value
0	1
1	0

Ordered pair 0, 1 as a matrix

index	value
0	0
1	1

An ordered foursome as a matrix.

index	value
0	2
1	365
2	3.1415...
3	6.6

In this presentation we will concern ourselves exclusively to one dimensional objects.

5. The Inner Product Of Ordered Pairs (Twosomes)

The inner product of two ordered Nsome results in a third ordered Nsome.

1. The product of the first number of both ordered Nsomes is the first number of the third ordered Nsome.

2. The product of the second number of both odered Nsomes is the second number of the third ordered Nsome.

3. The product of the nth number of both odered Nsomes is the nth number of the third ordered Nsome.

Example 1

ordered twosome 1

3, 4

ordered twosome 2

2, 6

ordered twosome 3

3x2 = 6

4x6 = 24

6, 24

Example 2

ordered twosome 1

1, 0

ordered twosome 2

0, 1

ordered twosome 3

1x0 = 0

0x1 = 0

0,0

We denote the inner product with the operator $\langle\rangle$. In the example above $\langle(3,4),(2,6)\rangle=(6,24)$

Note: In example 2, when the result of the inner product = 0,0, the first and second ordered twosomes are orthogonal if the inner product of the ordered twosomes with themselves is not 0,0.

The inner product of the ordered twosome 1,0 with itself is

$\langle(1,0),(1,0)\rangle=(1,0)$

and the inner product of the ordered twosome 0,1 with itself is

$\langle(0,1),(0,1)\rangle=(0,1)$

Since these inner products are not equal to 0,0, 1,0 and 0,1 are orthogonal ordered twosomes.

6. The Inner Product Of Matrices

The inner product of two ordered Nsomes can easily be applied to one dimensional matrices which are Nsomes with indices:

1. The product of the first number of both one dimensional matrices is the first number of the third one dimensional matrix.

2. The product of the second number of both one dimensional matrices is the second number of the third one dimensional matrix.

3. The product of the nth number of both one dimensional matrices is the nth number of the third one dimensional matrix.

Example 1

one dimensional matrix 1

index	value
0	3
1	4

one dimensional matrix 2

indexvalue

0 2

1 6

one dimensional matrix 3 is the inner product of matrix 1 and matrix 2

index value

0 3 x 2 = 6

1 4 x 6 = 24

Example 2

one dimensional matrix 1

indexvalue

0 1

1 0

one dimensional matrix 2

indexvalue

0	0
1	1

one dimensional matrix 3 is the inner product of matrix 1 and matrix 2

index	value
0	1x0 = 0
1	0x1 = 0

We denote the inner product with the operator $\langle \rangle$ with ordered Nsomes. In the example above except using square brackets for matrices:

$$\langle [3,4],[2,6] \rangle = [6,24]$$

Note: In example 2, when the result of the inner product = 0,0, the first and second one dimensional matrices are orthogonal if the inner product of the one dimensional matrices with themselves is not 0,0.

The inner product of the one dimensional matrix 1,0 with itself is

$$\langle [1,0],[1,0] \rangle = [1,0]$$

and the inner product of the one dimensional matrix 0,1 with itself is

$$\langle [0,1],[0,1] \rangle = [0,1]$$

Since these inner products are not equal to 0,0, 1,0 and 0,1 are orthogonal one dimensional matrices.

These ideas can be extended to matrices of one dimension and high cardinality, i.e. the number of elements in the matrix:

$$\langle [1,0,0],[1,0,0] \rangle = [1,0,0]$$

$$\langle [1,0,0,0],[1,0,0,0] \rangle = [1,0,0,0]$$

etc.

7. Linear Combinations Of Basis Matrices

A linear combinations of basis matrices is called synthesis.

A set of orthogonal matrices form a basis. A linear combination of basis matrices can generate other matrices of the same cardinality. A linear combination of basis matrices is defined as

$$c_x = a_0 \cdot b_0 + a_1 \cdot b_1 + \ldots a_{N-1} \cdot b_{N-1}$$

where

c_x = the resulting matrix

a_χ = coefficient matrix χ (chi)

b_χ = the 2 dimensional basis matrix, i.e. a collection of matrices $\chi = b(\chi,x)$

N = the cardinality of the matrix

χ = the χth basis matrix or coefficient

x = the xth element of the resulting matrix

Any one dimensional matrix of cardinality N can be created by a linear combination of basis vectors of cardinality N. A different way of stating this is that the basis spans (a term from matrix theory) the matrix space.

For example, matrix b is

0	1
1	0

χ=0, => $[x_{00}, x_{01}]$

$b_0 = [1,0]$

χ=1, => $[x_{10}, x_{11}]$

$b_1 = [0,1]$

$a_0 = 3.2$

$a_1 = 6$

$$c_x = a_0 \cdot b_0 + a_1 \cdot b_1 = 3.2 \cdot [1,0] + 6 \cdot [0,1] = [3.2,6]$$

Note that the resulting index is x

Note: The two indices for the b basis matrix makes it completely rigorous and to assist in understanding the extension from matrices to continuous functions in ℝ, the real numbers,:

8. Reversing Linear Combinations Of Basis Matrices

Generating the a coefficient matrix from a the synthesized matrix above is called analysis. This is done by simply taking the inner product of each constituent matrix of the basis matrix $b(\chi,x)$ with c_x, i.e. determine a_χ by taking the inner product of the corresponding basis function, $b_{\chi x}$ with c_x:

$$a_\chi = \langle c_x, b_{\chi x} \rangle$$

From the example above:

b_0 = [1,0]

c_x = [3.2,6]

$$\langle c_x, b_0 \rangle = \langle [1,0], [3.2,6] \rangle = 1 \cdot 3.2 + 0 \cdot 6 = 3.2 + 0 = 3.2 = a_0$$

$$\langle c_x, b_1 \rangle = \langle [0,1], [3.2,6] \rangle = 0 \cdot 3.2 + 1 \cdot 6 = 0 + 6 = 6 = a_1$$

We can also describe the coefficiants a_χ as the degree to which *c* resembles $b_{\chi x}$ or the degree to which $b_{\chi x}$ is present in c_x.

So the above should be written

$$a_\chi = \langle c_x, b_{\chi x} \rangle$$

Note also that the a_χ are scaler values, indexed by χ.

9. From Matrices To Functions

A function is a mapping from a domain to a range. Only one value in the range can be assigned a value in the domain. For example, the table shows the domain by the values of *x* and the range by the values of *F*(*x*). Each value of *x* is assigned a value *F*(*x*).

x	*F*(x)
0	2
1	365
2	3.1415
3	6.6

Table 1 Function Example 1

This is identical to the index and values of the matrix shown in section 4. If we equate the domain of the function with the index of the matrix, we can use the matrix properties of orthogonal basis and linear combinations to analyze functions using basis functions, the functional analog of basis matrices. The following figure shows how to illustrate a linear combination of the basis matrices [1,0] and [0,1] to create the function, illustrated in the

Cartresian coordinate plane (CCP), shown in Table 4. Note: There are two basis functions shown in Table 2 and Table 3, each of which is mapped on the x-coordinates 0

and 1 in the CCP.

x	*F*(x)
0	1
1	0

Table 2 Basis Function [01]

x	*F*(x)
0	0
1	1

Table 3 Basis Function ↔ [01]

x	*F*(x)
0	3.2
1	6

Table 4 Resulting Function ↔ 3.2 * [1,0] + 6 * [0,1]

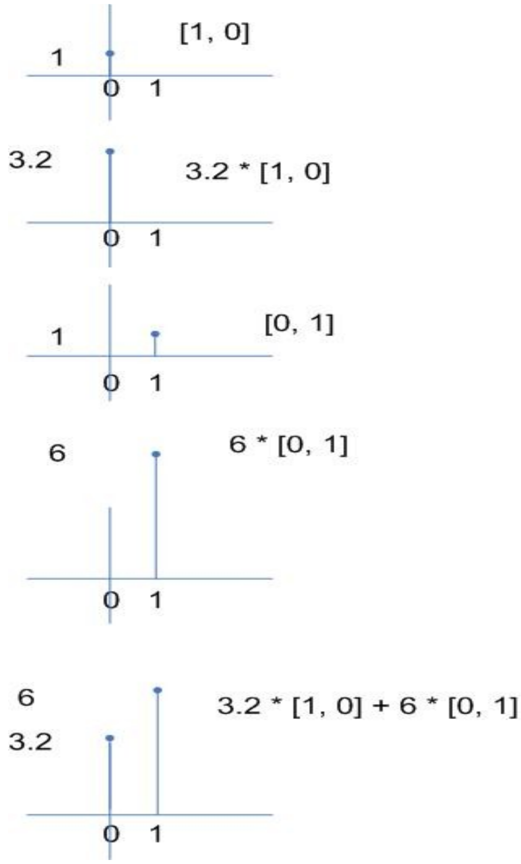


Figure 1 Function Example Using CCP

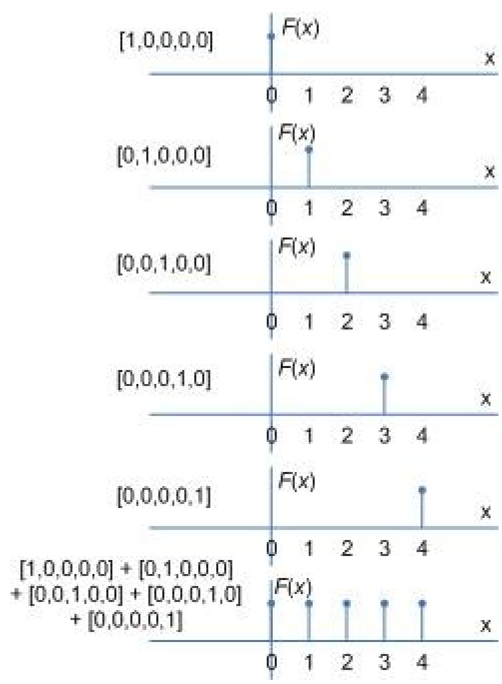


Figure 2 1 Dimensional Basis Matrices Of Cardinality 5

As shown in Figure 2, this idea can be extended to more basis functions (matrices), even an infinite number, extending to all x = 0, 1, 2, 3, ... All the the basis functions (matrices) still have the following properties:

1. The inner product of 2 different basis functions (matrices) is 0

2. The inner product of the basis function (matrix) with itself is 1

3. Any function whose domain is the natural numbers, ℕ, can be expressed as a linear combination of the basis functions (matrices).

4. We can determine a_χ by taking the inner product of the corresponding basis function, b_χ with c_x, the function.

10. Basis Functions With Negative Or Real Indices

Just as we can extend the basis functions (matrices) domain (indices) to all the natural numbers, when can extend it to the integers as well:

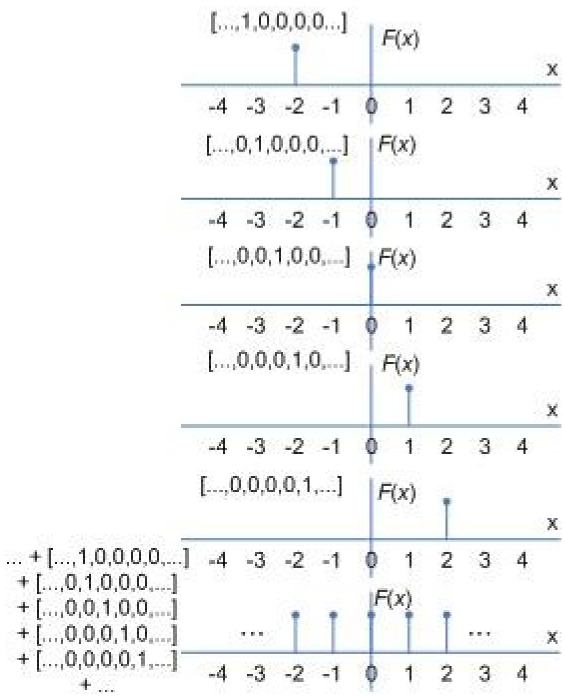


Figure 3 1 Dimensional Basis Matrices Of Countably Infinite Cardinality

As shown in Figure 3, all the the basis functions (matrices) still have the following properties:

1. The inner product of 2 different basis functions (matrices) is 0

2. The inner product of the basis function (matrix) with itself is 1

3. Any function whose domain is the integers, ℤ, can be expressed as a linear combination of the basis functions (matrices).

4. We can determine a_χ by taking the inner product of the corresponding basis function, b_χ with c_x, the function.

Just as we can extend the basis functions (matrices) domain (indices) to all the integers, when can extend it to the real numbers as well:

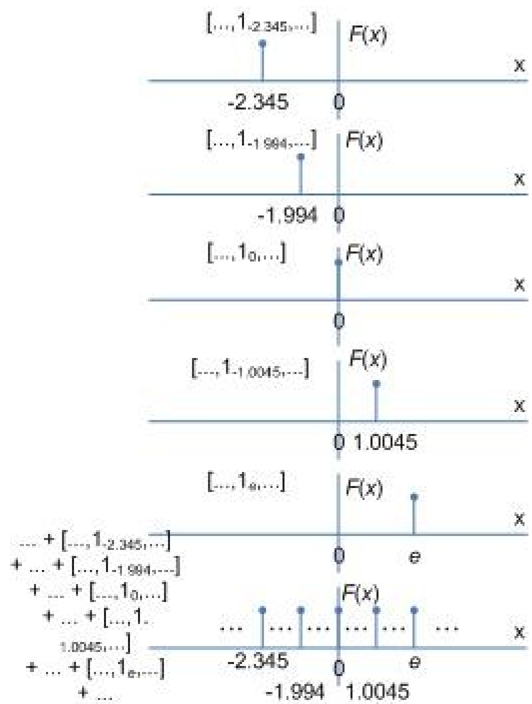


Figure 4 1 Dimensional Basis Matrices Of Uncountably Infinite Cardinality

As shown in Figure 4, all the the basis functions (matrices) still have the following properties:

1. The inner product of 2 different basis functions (matrices) is 0
2. The inner product of the basis function (matrix) with itself is 1
3. Any function whose domain is the real, ℝ, can be expressed as a linear combination of the basis functions (matrices).
4. We can determine a_χ by taking the inner product of the corresponding basis function, b_χ with c_x, the function.

11. Problems Of Functions Whose Domain is ℝ

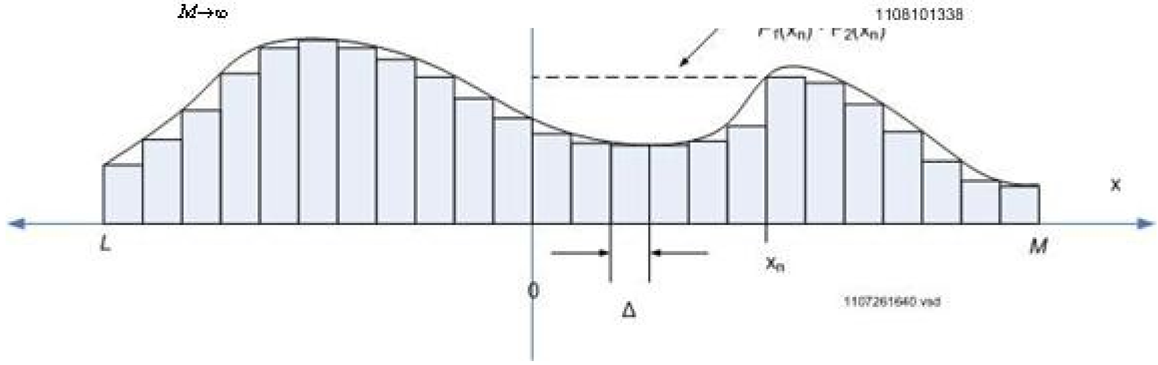
Obviously, the examples presented in the preceding sections become difficult when basis matix cardinality is very large, even infinite as in the case with basis functions whose domain is ℝ. However, integral calculus provides a means by which these functions can be manipulated quite easily. In fact, extending the ideas of orthogonality, the inner product and linear combinations of bases from simple ordered pairs to continuous functions in ℝ is one of the greatest advancements in mathematics that has occurred in the past 150 years.

12. Creating The Inner Product Of Two Functions In ℝ

The following equations and graph illustrate the very simple idea of the integral in calculus. Simple ideas, however, can have complex mathematical principles attached.

$$\lim_{\substack{\Delta\to 0\\ L\to -\infty\\ M\to\infty}} \sum_{x=L}^{x=M} F_1(x)F_2(x)\Delta = \int_{-\infty}^{\infty} F_1(x)F_2(x)dx$$

$$\lim_{\substack{\Delta\to 0\\ L\to -\infty\\ M\to\infty}} (\ldots + F1(x_{-1})F2(x_{-1}) + F1(x_0)F2(x_0) + F1(x_1)F2(x_1) + \ldots)\Delta$$



All the fancy math notation "simply" means the integral is a sum of rectangles that approximate the area under a funtion, the curve shown in figure , in this case the product of two functions $F_1(x)$ and $F_2(x)$, when $\Delta<0$ and the number of elements in the sum = (M-L)/Δ. The relationship between the number of "samples", N, i.e. rectangles and the distance between M and L is

$$N=\frac{M-L}{\Delta}$$

Also,

$$(M-L)\Delta=1 \Rightarrow N=\frac{1}{\Delta^2}$$

As $\Delta \rightarrow 0$, the number of samples increases exponentially until they reach an infinitely accurate approximation of the area under the curve = $F_1(x) \cdot F_2(x)$.

The integral of the product of two functions also has another property: As $\Delta \rightarrow 0$, it also reaches an infinitely accurate approximation of the inner product of the value of both functions for each real number! Note the ... + $F_1(x_n)\ F_2(x_n)$ + $F_1(x_{n+1})\ F_2(x_{n+1})$ +... terms above are exactly the same as the terms in the simple cases of matrix inner product presented in section 6 and converge to the inner product of each function for each real number.

Real Basis Functions

The basis function shown in Figure 4 is interesting as an illustration, but has no real practical purpose. There is very little information contained in a function which is 1 for 1 real number and 0 for all the rest. Are there any functions which met our set of requirements:

1. The inner product of 2 different basis functions (matrices) is 0

2. The inner product of the basis function (matrix) with itself is 1

3. Any function whose domain is the real, $\mathbb{R}$, can be expressed as a linear combination of the basis functions (matrices).

4. We can determine a_χ by taking the inner product of the corresponding basis function, $b_{\chi x}$ with c_x, the function.

Translating this to the language of calculus we obtain

$$1. \int_{-\infty}^{\infty} F_a(\chi, x) F_b(\chi, x) dx = 0$$

$$2. \int_{-\infty}^{\infty} F_a(\chi, x) F_a(\chi, x) dx = 1$$

$$3. f(\chi) = \int_{-\infty}^{\infty} F_b(\chi, x) F(x) dx$$

$$4. F(x) = \int_{-\infty}^{\infty} F_b(\chi, x) f(\chi) d\chi$$

The answer is yes:

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} F(x) e^{-2\pi x\xi} dx$$

Where $F_b(\chi,x) \leftrightarrow e^{-2\pi jx\xi}$

$\chi \leftrightarrow \xi$

13. Sinusoidal Basis Functions In $\mathbb{R}$

The rather formidable formulas and the end of the last section are actually structured simply. The complex expoential is, of course, Eulers astonishing discovery of the relationship with the transcendental constant $e = \ln^{-1}(1) =$

2.71828182845904523536028747135266249775724709369995...

and the sine and cosine functions.

The famous formula is

$$e^{jx} = \cos(x) + j sin(x)$$

$$j = \sqrt{-1}$$

By scaling the argument with x or t, which are real variables, to create a function we have

$$e^{j\omega x} = \cos(\omega x) + j sin(\omega x)$$

Where ω, i.e. frequency, corresponds to χ above

Summary

Analysis

$$a_\chi = \langle c_x, b(\chi, x)\rangle$$
$$\cdots$$
$$a_{-1} = \langle c_x, b(-1, x)\rangle = \ldots c_{-1}\cdot b_{-1,-1} + c_0\cdot b_{-1,0} + c_1\cdot b_{-1,1} + \ldots$$
$$a_0 = \langle c_x, b(0, x)\rangle = \ldots c_{-1}\cdot b_{0,-1} + c_0\cdot b_{0,0} + c_1\cdot b_{0,1} + \ldots$$
$$a_1 = \langle c_x, b(1, x)\rangle = \ldots c_{-1}\cdot b_{1,-1} + c_0\cdot b_{1,0} + c_1\cdot b_{1,1} + \ldots$$
$$\ldots$$
$$a_\chi = \sum_{x=\infty}^{\infty} c_x\cdot b_{\chi x} \Rightarrow$$
$$f(\chi) = \sum_{x=\infty}^{\infty} c_x\cdot b_{\chi x}\,\Delta x \Rightarrow$$
$$f(\chi) = \lim_{\Delta x\to 0} \sum_{x=\infty}^{\infty} c_x\cdot b_{\chi x}\,\Delta x$$
$$= \int_{\infty}^{\infty} f_c(x)\cdot f_{b\chi}(x)\,dx$$
$$= \int_{\infty}^{\infty} f_c(x)\cdot f_b(\chi, x)\,dx$$

The variable χ in the last equation creates a unique basis function, i.e. a set of unique basis functions.

Synthesis

$$c_x = \langle a_\chi, b(\chi, x)\rangle$$
$$\cdots$$
$$c_{-1} = \langle a_x, b(-1, x)\rangle = \ldots a_{-1}\cdot b_{-1,-1} + a_0\cdot b_{-1,0} + a_1\cdot b_{-1,1} + \ldots$$
$$c_0 = \langle a_x, b(0, x)\rangle = \ldots a_{-1}\cdot b_{0,-1} + a_0\cdot b_{0,0} + a_1\cdot b_{0,1} + \ldots$$
$$c_1 = \langle a_x, b(1, x)\rangle = \ldots a_{-1}\cdot b_{1,-1} + a_0\cdot b_{1,0} + a_1\cdot b_{1,1} + \ldots$$
$$\ldots$$
$$c_x = \sum_{x=\infty}^{\infty} a_\chi\cdot b_{\chi x} \Rightarrow$$
$$f(x) = \sum_{x=\infty}^{\infty} a_\chi\cdot b_{\chi x}\,\Delta\chi \Rightarrow$$
$$f(x) = \lim_{\Delta\chi\to 0} \sum_{x=\infty}^{\infty} a_\chi\cdot b_{\chi x}\,\Delta\chi$$
$$= \int_{\infty}^{\infty} f_a(\chi)\cdot f_{b\chi}(x)\,d\chi$$
$$= \int_{\infty}^{\infty} f_a(\chi)\cdot f_b(\chi, x)\,d\chi$$

The variable χ in the last equation creates a unique basis function, i.e. a set of unique basis functions.